Exercise 2

Use the *noise terms phenomenon* to solve the following Volterra integral equations:

$$u(x) = 6x + 3x^2 - \int_0^x xu(t)\,dt$$

[TYPO: The second term on the right should be $3x^3$ in order to get the answer at the back of the book.]

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x) &= 6x + 3x^3 - \int_0^x x \sum_{n=0}^{\infty} u_n(t)\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= 6x + 3x^3 - x\int_0^x [u_0(t) + u_1(t) + \cdots]\,dt \\
u_0(x) + u_1(x) + u_2(x) + \cdots &= \underbrace{6x + 3x^3}_{u_0(x)} + \underbrace{x\int_0^x [-u_0(t)]\,dt}_{u_1(x)} + \underbrace{x\int_0^x [-u_1(t)]\,dt}_{u_2(x)} + \cdots
\end{aligned}$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner.

$$\begin{aligned}
u_0(x) &= 6x + 3x^3 \\
u_1(x) &= x\int_0^x [-u_0(t)]\,dt = -x\int_0^x (6t + 3t^2)\,dt = -3x^3 - x^4 \\
&\vdots
\end{aligned}$$

The noise terms $\pm 3x^3$ appear in both $u_0(x)$ and $u_1(x)$. Cancelling $3x^3$ from $u_0(x)$ leaves $6x$. Now we check to see whether $u(x) = 6x$ satisfies the integral equation.

$$\begin{aligned}
6x &\stackrel{?}{=} 6x + 3x^3 - \int_0^x 6xt\,dt \\
6x &\stackrel{?}{=} 6x + 3x^3 - 3x^3 \\
6x &= 6x
\end{aligned}$$

Therefore,

$$u(x) = 6x.$$